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A $O(h^4)$ Cubic Spline Collocation Method For Quasilinear Parabolic Equations

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A modified version of the usual cubic spline collocation method is proposed and analyzed for quasilinear parabolic problems. Continuous time estimates of order $O(h^4)$ are obtained, via arguments based on certain discrete inner-products, for a uniform mesh and sufficiently smooth problems. Two types of collocation at the boundary are studied and shown to yield $O(h^4)$ and $O(h^{7/2})$ rates of convergence.

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A $O(h^4)$ Cubic Spline Collocation Method for Quasilinear Parabolic Equations *

1. Introduction. Consider the quasilinear parabolic equation

$$c(x,t,u)\, u_t - u_{xx} = f(x,t,u,u_x) \quad , \quad 0 < x < 1 \; , \quad 0 < t \leq T \tag{1.1a}$$

$$u(0,t) = b_o(t) \quad , \quad u(1,t) = b_1(t) \; , \quad 0 < t \leq T \; , \tag{1.1b}$$

$$u(x,0) = g(x) \quad , \quad 0 < x < 1 \quad . \tag{1.1c}$$

Let $\Delta = \{0 = x_o < x_1 < \cdot \cdot \cdot < x_N = 1\}$ be a partition of $I = [0,1]$, with $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$, and $h = \max\{h_i: 1 \leq i \leq N\}$. Then define

$$\Pi_k(J) = V:\{V \text{ is a polynomial of degree} < k \text{ on } J\}$$

and

$$\Pi_{k,\Delta} = \{V: V \in \Pi_k(I_i) \; , \quad 1 \leq i \leq N\} \; .$$

For $-1 \leq \ell \leq k-2$ let

$$S(\Delta,k,\ell) = \Pi_{k,\Delta} \cap C^{\ell}(I)$$

be the space of piecewise polynomials of degree $< k$ (order = k) on Δ with continuity ℓ . Note that $S(\Delta,k,\ell)$ has dimension $d[S(\Delta,k,\ell)] = kN - (\ell+1)(N-1)$. In this paper we shall be primarily concerned with $S_h \equiv S(\Delta,4,2)$, the usual cubic spline space on Δ .

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Recently Douglas and Dupont [10, 11] have studied collocation procedures for (1.1) based on the spaces $S(\Delta,k,1)$ for $k \geq 4$. Their main result is that collocation at the images of the k Gauss-Legendre points in each subinterval I_i yields uniform errors of order $O(h^k)$ and superconvergence results at the knots $\{x_i\}$ of order $O(h^{2k-4})$ if the solution of (1.1) $u \in H^{k+2}(I)$. These estimates (but not the analysis) are essentially the same as those of deBoor and Swartz [7] for ordinary differential equations. The analysis of [11] is based on certain discrete innerproducts as is the analysis presented in this paper.

Several authors [4 ,16 ,20] have studied collocation techniques for ordinary differential equations using smoother spaces $S(\Delta,k,\ell)$ with $\ell \geq 2$. The general result obtained is that the convergence rate is $O(h^{k-2})$, a suboptimal rate of convergence for such spaces. However, the procedures of Russell and Shampine [20] will provide $O(h^k)$ convergence (and superconvergence at the knots of order $O(h^{2k-6})$) for $k \geq 6$ if the collocation takes place at the images of the $k-2$ Lobatto points on each subinterval. Hence, it is expected that these procedures can be extended to parabolic problems through careful mimicing of the arguments in [11]. These procedures will be studied in a later paper.

In [8 , 19] cubic spline methods with $O(h^2)$ accuracy have been studied for linear versions of (1.1). Also, in [18] a cubic spline collocation procedure for the heat equation has been proposed (but not analyzed); for a particular explicit time discretization $O(h^4 + (\Delta t)^2)$ convergence obtains. However, this procedure is essentially the standard explicit finite difference method for the heat equation and the high accuracy does not readily generalize to more

difficult problems or other time discretizations.

In [2, 3] a variant of the usual cubic spline collocation method yielding $O(h^4)$ convergence rates for nonlinear ordinary differential and quasilinear parabolic equations was studied. In this paper we describe the high-order procedure and provide continuous time estimates for (1.1). Two types of boundary collocation will be considered, yielding uniform estimates of $O(h^4)$ and $O(h^{7/2})$ respectively. In a subsequent paper we shall investigate the effect of various boundary collocation techniques for high-order smooth spline approximations to (1.1).

It should be noted that the particular approximation used here is essentially the same as that of Daniel and Swartz [9] for two point boundary value problems. The procedures were developed independently, the derivation of [2] preceeding that of [9]. The finite-difference method discussed by Hirsh [13] can be interpreted as a cubic spline method, and as such, it is quite similar to the present technique.

This paper has four parts. In § 2 the basic notation of the paper is developed. Some discrete innerproducts for cubic splines are then defined and studied. The basic approximation technique used here is developed in § 3 from consideration of a simple two-point boundary value problem. In § 4 the main results are presented (Theorem 4.1).

2. Notation. We shall use the standard notation [14] for $L^p(I)$ spaces and Sobolev spaces $H^m(I)$. In particular, $||v||^2_{L^2(I)} = (v,v)$ with $(f,g) = \int_I f(x)g(x)dx$. Also, $H^1_o(I) = \{V \in H^1(I):\ v(0) = v(1) = 0\}$, with $||v||_{H^1_o(I)} = ||Dv||_{L^2(I)}$. Also, we use $W^m(I) = \{v: D^j v$ is abs. cont. $0 \le j \le m-1$, $D^m v \in L^\infty(I)\}$ with $||v||_{W^m(I)} = \sum_{\ell \le m} ||D^\ell v||_{L^\infty(I)}$.

The spaces $L^p[0,T;\ X]$ are defined as usual for normed linear spaces X [14].

If $v \in L^\infty(I)$ is defined on Δ , then write $v_i = v(x_i)$ and $\underset{\sim}{v} = (v_o, v_1, \cdots, v_N)^T$. Let $|||v|||_\infty \equiv ||\underset{\sim}{v}||_{\ell^\infty} = \max_{0 \le i \le N} |v_i|$.

Define the difference operators $\bar{\nabla} v_i = h_i^{-1}(v_i - v_{i-1})$, $\nabla v_i = \bar{\nabla} v_{i+1}$, and $\Delta^2 v_i = \nabla\bar{\nabla} v_i$. In case $v = v(x,y)$, denote the differences with respect to a particular variable as

$$(\bar{\nabla}_x v)(x_i,y) = h_i^{-1}(v(x_i,y) - v(x_{i-1},y)) \ ; \qquad \text{etc.}$$

In the following Δ is uniform; i.e., $\Delta = \{x_i = ih : 0 \le i \le N\}$. Then define the discrete innerproduct

$$[v,w] = \frac{h}{2} \sum_{i=1}^{N} (v_{i-1}w_{i-1} + v_i w_i)$$

with norm $|v|_2 = [v,v]^{\frac{1}{2}}$. Also, let $< v,w > = h \sum_{i=1}^{N-1} v_i w_i$ and $|v| = < v,v >^{\frac{1}{2}}$. Additionally, let $< v,w] = h \sum_{i=1}^{N} v_i w_i$.

Recall the summation by parts formula:

$$(2.1) \qquad < \nabla v, w > = - < v, \bar{\nabla} w] + v_N w_N - v_1 w_o \ .$$

The following results are easily established for cubic splines. Let

$$S_h^o = S_h \cap H_o^1(I) \quad .$$

<u>Lemma 2.1</u> If $v,w \in S_h^o$, then

$$\text{(2.2a)} \quad -< v'',w > = (v',w') + \frac{h^2}{12}(v'',w'') + \frac{h^4}{180}(v''',w''') - \frac{h^2}{6}B(v'',w') \quad ,$$

$$\text{(2.2b)} \quad -< v'' + \frac{h^2}{12}\Delta^2 v'' ,w > = (v',w') + \frac{h^4}{180}(v''',w''') - \frac{h^2}{12}B(v'',w') \ ,$$

$$\text{(2.2c)} \quad [v',w'] = (v',w') - \frac{h^4}{120}(v''',w''') + \frac{h^2}{12}\{B(v'',w') + B(w'',v')\} \ ,$$

$$\text{(2.2d)} \quad [v'',w''] = (v'',w'') + \frac{h^2}{6}(v''',w''') \quad ,$$

where

$$B(v,w) = v_N w_N - v_o w_o \quad .$$

Note that $v,w \in H_o^1(I)$ is not necessary for (2.2d).

<u>PROOF</u>: We prove (2.2a-b); the remaining results are similar. Recall the corrected trapezoidal rule

$$\int_c^d \phi(x)\,dx = \frac{d-c}{2}[\phi(c) + \phi(d)] - \frac{(d-c)^2}{12}\left[\phi'\Big|_c^d\right] + \frac{(d-c)^5}{720}\phi^{(4)}(\xi) \ , \ \xi \in (c,d) \ .$$

Applying this rule one interval at a time and summing yields

$$(v'',w) = [v'',w] - \frac{h^2}{12} < \overline{\nabla} v'',\overline{\nabla} w \,] + \frac{h^4}{180}(v''',w''') - \frac{h^2}{12}B(v'',w')$$

for all $v,w \in S_h$. Summation by parts and $v,w \in H_o^1(I)$ imply that

$$\text{(2.3)} \quad (v'',w) = [v'',w] + \frac{h^2}{12} < \Delta^2 v'',w > + \frac{h^4}{180}(v''',w''') - \frac{h^2}{12}B(v'',w')$$

which is (2.2b). It is easy to show that

$$\text{(2.4)} \quad (v'',w'') = B(v'',w') + < \Delta^2 v'',w > \ ;$$

hence, (2.2a) follows from (2.3) and (2.4).

To apply these results, we need the inverse relations (not assumptions for splines).

<u>Lemma 2.2</u> [21] If $v \in \Pi_{k,\Delta}$ with $h/\min\limits_{1\le i\le N} h_i \le \sigma$, then

(2.5) $$||Dv||_{L^q(I)} \le Ch^{-1}||v||_{L^q(I)} ,$$

(2.6) $$||v||_{L^q(I)} \le Ch^{\frac{1}{q}-\frac{1}{p}}||v||_{L^p(I)} , \quad 1 \le p \le q \le \infty .$$

Let $|v|_\partial \equiv \max\limits_{x=0,1} |v(x)|$. Then for $v \in S_h^o$,

(2.7) $$|B(v'',v')| \le 2 |v''|_\partial |v'|_\partial \le Ch^{-2} ||v||^2_{H^1_o(I)} .$$

Hence, by (2.2c)

(2.8) $$|v|^2 + |v'|^2 \le C||v||^2_{H^1_o(I)} .$$

Since $B(v'',v')$ is not definite, the left sides of (2.2a) and (2.2b) (with $v = w$) are not norms equivalent to the $H^1_o(I)$ norm on S_h^o in general. Of course, if v is periodic, $B(v'',v') = 0$ and the forms in question are actually equivalent to the $H^1_o(I)$ norm.

It is not generally true that $|v|$ and $||v||_{L^2(I)}$ are equivalent; however, it is the case that

(2.9) $$|v| \le C||v||_{L^2(I)} .$$

It is true that

(2.10) $$C_1 ||v||^2_{L^2(I)} \le |v|^2 + h^5|v''|^2_\partial \le C_2||v||^2_{L^2(I)} .$$

For this note first of all that the exponent of h is correct by Lemma 2.2. Also, if

$$|v|^2 + h^5 |v''|_\partial^2 = 0 \quad ,$$

then

$$v_i = 0 \quad , \qquad 0 \le i \le N$$

and

$$v_i'' = 0 \quad , \qquad i = 0, N$$

It is then clear that $v \equiv 0$. Hence, $|||v||| \equiv \left(|v|^2 + h^5 |v''|_\partial^2 \right)^{1/2}$ is a norm on S_h^o and (2.10) follows.

We shall also use the following notation

$$< v, w >_A \equiv < v + \frac{h^2}{12} \Delta^2 v \, , \, w > \quad ,$$

for mesh functions v and w .

3. <u>A Two-Point Boundary Value Problem</u>. In this section we consider a cubic spline approximation to the two-point boundary value problem

(3.1a) $$u''(x) = f(x) \quad , \quad x \in I$$

(3.1b) $$u(0) = b_o \quad , \quad u(1) = b_1 \quad .$$

It is well-known [4] that collocation at the knots $\{x_i\}_{i=0}^N$ in S_h ; i.e., finding $U_c \in S_h$ such that

(3.2a) $$U_c''(x_i) = f(x_i) \qquad , \quad 0 \le i \le N$$

(3.2b) $$U_c(0) = b_o \quad , \quad U_c(1) = b_1 \quad ,$$

has a convergence rate $0(h^2)$ (and no better) in general. However, defining $U \in S_h$ by

(3.3a) $$U''(x_i) = f(x_i) - \frac{h^2}{12} f''(x_i) \quad , \quad 0 \le i \le N$$

(3.3b) $$U(0) = b_o \quad , \quad U(1) = b_1 \quad ,$$

leads to the following results.

<u>Theorem 3.1</u> Suppose $u \in W^6(I)$ is the solution of (3.1) and $U \in S_h$ is defined by (3.3). Then the following estimates hold for $e = u - U$:

(3.4a) $$||D^j e||_{L^\infty(I)} \le Ch^{4-j}||u||_{W^6(I)} \quad , \quad 0 \le j \le 3 \quad .$$

The following superconvergence results are also valid for $1 \le i \le N$:

(3.4b) $$|e_i'| \le Ch^4||u||_{W^6(I)} \quad ,$$

(3.4c) $$|e_{i-\frac{1}{2}}'| \le Ch^4||u||_{W_\infty^6(I)} \quad ,$$

(3.4d) $$|e''(\xi_{ij})| \le Ch^3||D^5u||_{L^\infty(I_i)} \quad ; \quad j = 1 \, , \, 2 \, ,$$

(3.4e) $$|e_{i-\frac{1}{2}}'''| \le Ch^2||D^5u||_{L^\infty(I_i)} \quad ,$$

where

(3.4f) $$x_{i-\frac{1}{2}} = (x_{i-1} + x_i)/2$$

and

(3.4g) $$\xi_{ij} = x_{i-\frac{1}{2}} + (-1)^j \frac{h}{2\sqrt{3}} , \quad j = 1,2 .$$

Additionally, if $u \in W^{6+k}(I)$ for $0 \le k \le 2$, then

(3.4h) $$U''_i + \frac{h^2}{12} (\Delta^2 U'')_i = U''_i + O\left(h^{4+k} ||D^{6+k} u||_{L^\infty(I)}\right), \quad \text{for} \quad 1 \le i \le N-1 .$$

<u>Proof</u>: Expand e'' about x_{i-1} on I_i to obtain for $0 \le \tau \le h$:

(3.5) $$e''(x_{i-1} + \tau) = \left(\frac{h^2}{12} - \frac{\tau h}{2} + \frac{\tau^2}{2}\right) f''_{i-1} + \left(\frac{\tau^3}{6} - \frac{\tau h^2}{12}\right) f'''_{i-1}$$
$$+ \; O\left(h^4 ||f^{iv}||_{L^\infty(I_i)}\right) .$$

Then it is straightforward that

(3.6) $$\left| \int_{I_i} e''(x) p_2(x) \, dx \right| \le Ch^5 \, ||u||_{W^6(I_i)} , \quad 1 \le i \le N ,$$

for any $p_2 \in \Pi_{2,\Delta}$ bounded independently of h .

Let $G_o(x;\xi)$ be the Green's function for $v'' = g$ on I subject to $v(0) = v(1) = 0$, and define $G_1(x;\xi) = \left(\frac{\partial}{\partial x} G_o\right)(x;\xi)$. Since $G_o(x; \cdot) \in W^1(I)$ and $G_j(x_i; \cdot) \in \Pi_{2,\Delta}$ for $0 \le i \le N$, $j = 0,1$, we have

(3.7) $$\begin{aligned} |D^j e(x_i)| &= \left| \int_I G_j(x_i ; \xi) e''(\xi) \, d\xi \right| \\ &\le \sum_{n=1}^{N} \left| \int_{I_n} G_j(x_i ; \xi) e''(\xi) \, d\xi \right| \\ &\le Ch^5 \sum_{n=1}^{N} ||u||_{W^6(I_n)} \\ &\le Ch^4 \, ||u||_{W^6(I)} . \end{aligned}$$

The stability results of [22] and (3.7) imply

$$(3.8) \qquad \|D^j e\|_{L^\infty(I)} \le Ch^{4-j} \|u\|_{W^6(I)} \quad , \quad 0 \le j \le 3 \ .$$

Integration of (3.5) from x_{i-1} to $x_{i-\frac{1}{2}}$ yields (3.4c). Estimates (3.4d-e) follow immediately from (3.5).

Since for $1 \le i \le N-1$ and $0 \le k \le 2$

$$(3.9) \qquad (\Delta^2 U'')_i = [\Delta^2(f - \frac{h^2}{12} f'')]_i = f_i'' + O\left(h^{2+k} \|D^{4+k} f\|_{L^\infty(I)}\right) \quad ,$$

estimate (3.4h) is established, and the proof is complete.

We now consider defining $W \in S_h$ by (3.4h) neglecting the $O\left(h^{4+k} \|D^{6+k} u\|_{L^\infty(I)}\right)$ terms. More precisely, define $W \in S_h$ by

$$(3.10a) \qquad W''(x) = f(x) - \frac{h^2}{12} f''(x) \quad , \quad x = 0,1 \quad ,$$

$$(3.10b) \qquad W_i'' + \frac{h^2}{12} (\Delta^2 W'')_i = f_i \quad , \quad 1 \le i \le N-1$$

$$(3.10c) \qquad W(0) = b_o \quad , \quad W(1) = b_1 \ .$$

The following results then obtain .

<u>Corollary 3.2</u> Let u , U be as in Theorem 3.1. Define $W \in S_h$ by (3.10). Let $z = U-W$ and $\tilde{e} = z + e = u - W$. Then

$$(3.11a) \qquad \|D^j z\|_{L^\infty(I)} \le Ch^{4+k} \|D^{6+k} u\|_{L^\infty(I)} \quad , \quad 0 \le j \le 2 \ .$$

$$(3.11b) \qquad \|z'''\|_{L^\infty(I)} \le Ch^{3+k} \|D^{6+k} u\|_{L^\infty(I)} \quad ,$$

if $u \in W^{6+k}(I)$, $0 \le k \le 2$.

Furthermore, all the inequalities of (3.4) hold with $\tilde{e}$ replacing e

and the norm of u on the right side of each inequality being changed to $||u||_{W^6(I)}$.

<u>Proof</u>: Equations (3.10a-b) yield an $(N+1 \times N+1)$ linear system for W_i'' ,

$$A\,\underset{\sim}{W}'' = \overline{\underset{\sim}{f}} \equiv \underset{\sim}{f} - \frac{h^2}{12}\,(f_o'' ,0,\cdots,0,f_N'')^T \tag{3.12}$$

where

$$A = \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 & 0 & 0 & \cdots & & 0 \\ 1 & 10 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 10 & 1 & 0 & \cdots & & 0 \\ & & & & & & & \\ 0 & & \cdots & & 0 & 1 & 10 & 1 \\ 0 & & \cdots & & 0 & 0 & 0 & 12 \end{bmatrix}$$

Since A is diagonally dominant with $||A^{-1}||_\infty \leq \frac{3}{2}$, $\underset{\sim}{W}''$ (hence W) is uniquely defined by (3.11). By (3.4h) $\underset{\sim}{U}''$ satisfies

$$A\,\underset{\sim}{U}'' = \overline{\underset{\sim}{f}} + \underset{\sim}{\delta} , \tag{3.14}$$

where

$$\underset{\sim}{\delta} = (0,\delta_1 ,\cdots, \delta_{N-1},0)^T$$

and

$$|\delta_i| \leq Ch^{4+k}\,||u^{6+k}||_{L^\infty(I)} , \quad 1 \leq i \leq N-1 .$$

Subtracting (3.12) from (3.14) yields

$$A\,\underset{\sim}{z}'' = \underset{\sim}{\delta} , \tag{3.15}$$

from which

$$(3.16)\qquad |||z''|||_\infty = ||\underset{\sim}{z}''||_{\ell^\infty} \le \frac{3}{2}||\underset{\sim}{\delta}||_{\ell^\infty} \le Ch^{4+k}||D^{6+k}u||_{L^\infty(I)} \quad .$$

Since z is piecewise linear

$$(3.17)\qquad ||z''||_{L^\infty(I)} \le |||z''|||_\infty \le Ch^{4+k}||D^{6+k}u||_{L^\infty(I)} \quad .$$

Finally, the Green's function representation leads to for $j = 0,1$

$$(3.18)\qquad |D^j z(x)| = |\int_I G_j(x\ ;\ \xi)z''(\xi)\ d\xi| \le ||G_j(x\ ;\ \cdot)||_{L^\infty(I)}\ ||z''||_{L^\infty(I)} \quad ,$$

which along with (3.17) establishes (3.11a). The remainder of the Corollary follows from homogeneity in h and the triangle inequality applied to $\tilde{e} = z + e$.

4. Continuous Time Estimates. In this section we consider the continuous-in-time approximation to the solution of (1.1) by cubic spline methods. Define $U : [0,T] \to S_h$ by

(4.1a) $$\left(U_{xx} + \frac{h^2}{12} c(U)U_{xxt}\right)(x,t) = \left(A_h(U)U_t - B_h(U)U_{xt} - F_h(U,U_x)\right)(x,t), \quad x = 0,1 \quad ,$$

(4.1b) $$\left(c(U)U_t - [U_{xx} + \frac{h^2}{12} \Delta_x^2 U_{xx}]\right)(x_i,t) = \left(f(U,U_x)\right)(x_i,t) \quad , \qquad 1 \le i \le N-1 \ ,$$

(4.1c) $$U(0,t) = b_o(t), \ U(1,t) = b_1(t) \quad ,$$

(4.1d) $$U(x,0) - g(x) = \text{"small"} \quad , \quad x \in I \ .$$

In (4.1a) we have used (supressing (x,t))

(4.2a) $$A_h(\phi) = c(\phi) - \frac{h^2}{12} D_h^2 [c(\phi)]$$

(4.2b) $$B_h(\phi) = \frac{h^2}{6} D_h^1 [c(\phi)]$$

(4.2c) $$F_h(\phi,\phi_x) = f(\phi,\phi_x) - \frac{h^2}{12} D_h^2 [f(\phi,\phi_x)]$$

where

(4.2d) $$D_h^1[\psi](x) = \begin{cases} \frac{3}{2} (\overline{\nabla}_x \psi)_1 - \frac{1}{2} (\overline{\nabla}_x \psi)_2 \quad , & x = 0 \\ \\ \frac{3}{2} (\overline{\nabla}_x \psi)_N - \frac{1}{2} (\overline{\nabla}_x \psi)_{N-1} \ , & x = 1 \quad , \end{cases}$$

and

$$(4.2e)\qquad D_h^2[\psi](x) = \begin{cases} 2(\Delta_x^2\psi)_1 - (\Delta_x^2\psi)_2 , & x = 0 \\ 2(\Delta_x^2\psi)_{N-1} - (\Delta_x^2\psi)_{N-2} , & x = 1 . \end{cases}$$

Note that for $\psi \in W^{2+k}(I)$, $0 \le k \le 2$

$$(4.3a)\qquad R_h^1(\psi) \equiv D_x\psi - D_h^1[\psi] = O\left(h^k ||D_x^{k+1}\psi||_{L^\infty(I)}\right)$$

$$(4.3b)\qquad R_h^2(\psi) \equiv D_x^2\psi - D_h^2[\psi] = O\left(h^k ||D_x^{k+2}\psi||_{L^\infty(I)}\right) .$$

In the following we shall assume that c and f are smooth functions of their arguments with c subject to the bounds

$$(4.4)\qquad 0 < m \le c(x,t,\phi) \le M < \infty$$

$$x \in I , \; t \in [0,T] \quad \text{and} \quad \phi \in \mathcal{R} = (-\infty,\infty) .$$

The choice of this particular approximation is motivated by the results of §3, specifically Corollary 3.2. Later $U(x,0)$ will be chosen to provide the desired $O(h^4)$ convergence rates. It is also possible to use a different collocation procedure at the boundary, namely

$$(4.1a)'\qquad \left(c(U)U_t - U_{xx}\right)(x,t) = f(x,t,U,U_x) , \quad x = 0,1 ;$$

however, the analysis here will provide only $O(h^{7/2})$ rates of convergence.

The analysis of (4.1) will proceed along the same lines as that in [11] and will employ the discrete inner products of §2 . Before beginning the error analysis, we establish the existence and uniqueness of the solution of

(4.1). For this, we consider the equivalent matrix formulation based on the B-spline basis $\{V_1, V_2, \ldots, V_{N+3}\}$ on the knot set

$$\left\{\, 0 = \tau_i \ (1 \le i \le 4),\ \tau_{4+i} = x_i \ (1 \le i \le N-1),\ \tau_{N+3+i} = 1\ ,\ (1 \le i \le 4) \right\};$$

see [5, 6]. Let $U(x,t) = \sum_{j=1}^{N+3} \alpha_j(t) V_j(x)$. Then (4.1) becomes

(4.5a) $$\mathcal{C}(\alpha)\underset{\sim}{\alpha}'(t) - \mathcal{A}\underset{\sim}{\alpha} = \mathcal{F}(\underset{\sim}{\alpha}) \quad , \quad t \in (0,T]$$

(4.5b) $$\alpha_1(t) = b_0(t) \quad , \quad \alpha_{N+3}(t) = b_1(t)\ , \ t \in (0,T]$$

(4.5c) $$\underset{\sim}{\alpha}(0) = \text{given}\ ,$$

where $\underset{\sim}{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_{N+3}(t))^T$, and for $1 \le j \le N+3$

(4.6a) $$[\mathcal{C}(\alpha)]_{ij} = \begin{cases} c\left(\sum_k \alpha_k V_k(x_i)\right) V_j(x_i) & , \quad 1 \le i \le N-1 \\ \frac{h^2}{12} c\left(\sum_k \alpha_k V_k(x_i)\right) V_j''(x_i) - A_h\left(\sum_k \alpha_k V_k(x_i)\right) V_j(x_i) & \\ \quad + B_h\left(\sum_k \alpha_k V_k(x_i)\right) V_j'(x_i) & , \quad i = 0,N \end{cases}$$

(4.6b) $$[\mathcal{A}]_{ij} = \begin{cases} V_j''(x_i) + \frac{h^2}{12} (\Delta_x^2 V_j'')(x_i) & , \quad 1 \le i \le N-1 \\ -V_j''(x_i) & , \quad i = 0,N \end{cases}$$

(4.6c) $$[\mathcal{F}(\alpha)]_i = \begin{cases} f\left(\sum_k \alpha_k V_k(x_i)\ ,\ \sum_k \alpha_k V_k'(x_i)\right) & , \quad 1 \le i \le N-1 \\ -F_h\left(\sum_k \alpha_k V_k(x_i)\ ,\ \sum_k \alpha_k V_k'(x_i)\right) & , \quad i = 0,N \end{cases}$$

The assumption that f is Lipschitz continuous with respect to its last two arguments implies that $\mathcal{F}(\alpha)$ is likewise Lipschitz continuous. Thus, the local existence (in time) of the solution to (4.1) will be established in case

$$\mathcal{C}(\alpha)\underset{\sim}{\beta} = 0 \quad \text{and} \quad \beta_1 = \beta_{N+3} = 0 \tag{4.7}$$

implies that $\underset{\sim}{\beta} = 0$. Let $\phi(x) = \sum_{k=1}^{N+3} \alpha_k V_k(x)$ and suppose that $\psi(x) = \sum_{k=1}^{N+3} \beta_k V_k(x)$ with $\underset{\sim}{\beta}$ satisfying (4.7). Then

$$\psi(x_i) = 0 \quad , \quad 0 \le i \le N \ , \tag{4.8a}$$

and

$$\{c(\phi)\psi_{xx} + 2D_h^1 [c(\phi)]\psi_x\}(x_i) = 0 \quad , \quad i = 0,N \ . \tag{4.8b}$$

By the standard cubic spline identities [15], (4.8) is equivalent to

$$\psi_{xx}(x_{i-1}) + 4\psi_{xx}(x_i) + \psi_{xx}(x_{i+1}) = 0 \quad , \quad 1 \le i \le N-1 \ , \tag{4.9a}$$

$$\{c(\phi) - \frac{2h}{3} D_h^1 [c(\phi)]\}(0)\psi_{xx}(0) - \frac{h}{3} D_h^1 [c(\phi)](0)\psi_{xx}(x_1) = 0 \ , \tag{4.9b}$$

$$\{c(\phi) + \frac{2h}{3} D_h^1 [c(\phi)]\}(1)\psi_{xx}(1) + \frac{h}{3} D_h^1 [c(\phi)](1)\psi_{xx}(x_{N-1}) = 0 \ . \tag{4.9c}$$

Assuming that

$$\left|\frac{\partial}{\partial x} [c(x,t,\phi)]\right| \le L \quad \text{for} \quad x \in I \ , \ t \in (0,T], \ \phi \in R \ , \tag{4.10}$$

we find from (4.3a) that $\{D_h^1 [c(\phi)]\}(x)$ is bounded for $x = 0,1$. Thus, for sufficiently small h, (4.9) corresponds to a diagonally dominant, homogeneous linear system for $\psi_{xx}(x_i)$. Hence, $\psi_{xx} \equiv 0$ and $\underset{\sim}{\beta} \equiv 0$.

Lemma 4.1 If (4.10) holds then for h sufficiently small there exists a unique $U \in S_h$ solving (4.1) for $t \in (0,T]$.

Lemma 4.2 If

(4.10a)' $\quad \left|\frac{\partial}{\partial t}[c(x,t,\phi)]\right| \leq L \quad , \quad x \in I \, , \, t \in (0,T] \, , \, \phi \in R \, ,$

and

(4.10b)' $\quad \left|\frac{\partial}{\partial t}[f(x,t,\phi,\psi)]\right| \leq L \quad , \quad x \in I \, , \, t \in (0,T], \, \phi,\psi \in R,$

then for h sufficiently small, there exists a unique solution $U_h \in S_h$ of (4.1) with (4.1a)' replacing (4.1a) for $t \in (0,T]$.

Proof: Similar to the above. Just differentiate (4.1a)' with respect to t to obtain an analogue of (4.9) which is diagonally dominant for h small enough.

We now turn to the convergence analysis of (4.1). Note that (4.1b) is equivalent to the discrete Galerkin formulation

(4.11) $\quad < c(U)U_t, V > - < U_{xx}, V >_A = < f(U,U_x), V > \, , \quad V \in S_h \quad .$

For the analysis define the comparison function $W : [0,T] \to S_h$ by

(4.12a) $\quad W_{xx}(x,t) = u_{xx}(x,t) - \frac{h^2}{12} u_{xxxx}(x,t) \quad , \quad x = 0,1 \quad ,$

(4.12b) $\quad (W_{xx} + \frac{h^2}{12} \Delta_x^2 W_{xx})(x_i,t) = u_{xx}(x_i,t) \quad , \quad 1 \leq i \leq N-1 \quad ,$

(4.12c) $\quad W(0,t) = b_0(t) \quad , \quad W(1,t) = b_1(t)$

Note that Corollary 3.2 implies that W (W_t) is a $O(h^4)$ approximation to u (u_t) .

Let $z = W - U \in S_h^o$, $\tilde{e} = u - W$, and $e = z + \tilde{e} = u - U$. Our plan is to estimate z in terms of $\tilde{e}$ and then to bound e using the bounds on z and the triangle inequality. In the following analysis, we shall often require the inequality $ab \leq \varepsilon a^2 + (1/4\varepsilon)b^2$ for $a,b \leq 0$, any $\varepsilon > 0$.

From (4.12b) and (1.1) we find

$$(4.13)\qquad \begin{aligned} &< c(W)W_t,V > - < W_{xx},V >_A = - < c(W)\,\tilde{e}_t,V > \\ &\qquad + < [c(W) - c(u)]u_t,V > + < f(u,u_x),V > , \quad V \in S_h \ . \end{aligned}$$

Subtract (4.11) from (4.13) and apply the assumed smoothness of c and f to obtain

$$(4.14)\qquad \begin{aligned} < c(U)z_t,V > - < z_{xx},V >_A &= < [c(W) - c(U)]W_t,V > - < c(W)\tilde{e}_t,V > \\ &\quad + < [c(W) - c(u)]u_t,V > \\ &\quad + < f(u,u_x) - f(U,U_x), V > \\ &= < c_u^* zW_t,V > - < c(W)\tilde{e}_t,V > \\ &\quad - < c_u^* \tilde{e}\, u_t,V > . \\ &\quad + < f_u^* \tilde{e} + f_{u_x}^* \tilde{e}_x,V > \\ &\quad + < f_u^* z + f_{u_x}^* z_x ,V > \quad , \end{aligned}$$

where the partial derivatives c_u, f_u, f_{u_x} are evaluated as required by the mean value theorem. Now use Cauchy-Schwarz, the boundedness of the derivatives of c and f , and the trivial inequality mentioned above with $V = z_t$:

$$(4.15)\qquad < c(U)z_t,z_t > - < z_{xx},z_t >_A \leq C(|z|^2 + |z_x|^2 + |T\tilde{e}|^2) + \delta|z_t|^2 , \quad \delta > 0 .$$

Here

$$(4.16)\qquad |T\tilde{e}|^2 = |\tilde{e}|^2 + |\tilde{e}_x|^2 + |\tilde{e}_t|^2 \ .$$

If

$$(4.17)\qquad u, u_t \in L^2[0,T;W^6(I)] \ ,$$

Corollary 3.2 implies that

$$(4.17b)\qquad \int_0^t |T\tilde{e}|^2 d\tau \le Ch^8 \left(||u||^2_{L^2[0,T;W^6(I)]} + ||u_t||^2_{L^2[0,T;W^6(I)]} \right) \ .$$

Choose δ so that $\eta = m-\delta > 0$ and use (2.8) to obtain

$$(4.18)\qquad \eta|z_t|^2 - < z_{xx}, z_t >_A \le C(||z||^2_{H^1_o(I)} + |T\tilde{e}|^2) \ .$$

To complete the estimate, it is necessary to consider the boundary terms (4.1a). A straight-forward computation using (4.12a) and (4.3) yields

$$(4.19)\qquad \begin{aligned} \{W_{xx} + \frac{h^2}{12} c(W) W_{xxt}\}(x,t) &= \{A_h(u)u_t - B_h(u)u_{xt} - F_h(u,u_x)\}(x,t) \\ &\quad - \frac{h^2}{12}\{c(u)\tilde{e}_{xxt}\}(x,t) \\ &\quad + R_h(x,t) \quad , \quad x = 0,1 \quad , \end{aligned}$$

where

$$(4.20a)\qquad R_h(x,t) = -\frac{h^2}{12}\{R_h^2\ [c(u)] - 2R_h^1\ [c(u)] - R_h^2\ [f(u,u_x)]\}(x,t) \ .$$

Note that if $\quad I_h = [0,x_3] \cup [x_{N-3},1] \quad ,$

$$(4.20b)\qquad \bar{c}(x,t) = c(x,t,u) \in W^4(I_h)$$

and

$$(4.20c)\qquad \bar{f}(x,t) = f(x,t,u,u_x) \in W^4(I_h) \quad , \quad t \in (0,T]$$

then by (4.3)

$$(4.20d)\qquad |R_h|_\partial(t) \le Ch^4 \left(||D_x^4 \bar{c}||_{L^\infty(I_h)} + ||D_x^4 \bar{f}||_{L^\infty(I_h)} \right)(t) \ .$$

In the sequel, we assume that

(4.20e) $$K_h \equiv ||D_x^4 \bar{c}||_{L^\infty[0,T;L^\infty(I_h)]} + ||D_x^4 \bar{f}||_{L^\infty[0,T;L^\infty(I_h)]} < \infty \ .$$

Subtract (4.1a) from (4.19) to obtain

(4.21)
$$\begin{aligned} z_{xx} + \frac{h^2}{12} c(u) z_{xxt} &= \left(A_h(u) - A_h(U)\right) u_t \\ &- \left(B_h(u) - B_h(U)\right) u_{xt} \\ &- \left(B_h(U) \ (z_{xt} + \tilde{e}_{xt})\right) \\ &- (F_h(u, u_x) - F_h(u, U_x)) \\ &- (F_h(u, U_x) - F_h(U, U_x)) \\ &- \frac{h^2}{12} c(u) \ \tilde{e}_{xxt} + R_h \ . \end{aligned}$$

We now estimate the terms on the right side of (4.21). The treatment of all but one of the terms is somewhat rough.

(4.22a)
$$\begin{aligned} |A_h(u) - A_h(U)|_\partial &\le |A_h(u) - A_h(W)|_\partial + |A_h(W) - A_h(U)|_\partial \\ &= \frac{h^2}{12}\left(|D_h^2 \ [c(u) - c(W)]|_\partial + |D_h^2 \ [c(W) - c(U)|_\partial \right) \\ &= \frac{h^2}{12}\left(|D_h^2 \ [c_u^* \ \tilde{e}]|_\partial + |D_h^2 \ [c_u^* \ z]|_\partial \right) \\ &\le C(|||\tilde{e}|||_\infty + |||z|||_\infty) \end{aligned}$$

Similarly,

(4.22b) $$|B_h(u) - B_h(U)|_\partial \le Ch(|||\tilde{e}_x|||_\infty + |||z_x|||_\infty)$$

(4.22c) $$|F_h(u, u_x) - F_h(u, U_x)|_\partial \le C(|||\tilde{e}_x|||_\infty + |||z_x|||_\infty)$$

and

(4.22d) $$|F_h(u, U_x) - F_h(U, U_x)|_\partial \le C(|||\tilde{e}|||_\infty + |||z|||_\infty) \ .$$

Recalling (4.10) and (4.3a) with $k = 0$,

$$|B_h(U)|_\partial \leq Ch^2 \quad .$$

Thus,

$$|B_h(U)(z_{xt} + \tilde{e}_{xt})|_\partial \leq Ch^2(|z_{xt}|_\partial + |\tilde{e}_{xt}|_\partial) \quad . \tag{4.22e}$$

Now multiply (4.21) by z_{xxt} , integrate in t , and apply (4.22) to obtain

$$\begin{aligned} \frac{1}{2}|z_{xx}|_\partial^2(t) + \mu h^2 \int_0^t |z_{xxt}|_\partial^2 d\tau \leq \; & Ch^{-2}\int_0^t \Big(|||z|||_\infty^2 + |||z_x|||_\infty^2 + h^4|z_{xt}|_\partial^2 \\ & + |T\tilde{e}|_\partial^2 + |R_h|_\partial^2 \Big) d\tau \\ & + \frac{1}{2}|z_{xx}|_\partial^2(0) \quad , \end{aligned} \tag{4.23}$$

where $\mu > 0$

$$|Te|_\partial^2 = |||\tilde{e}|||_\infty^2 + |||\tilde{e}_x|||_\infty^2 + h^4\left(|\tilde{e}_{xt}|_\partial^2 + |\tilde{e}_{xxt}|_\partial^2\right) \tag{4.24a}$$

$$\int_0^t |T\tilde{e}|_\partial^2 d\tau \leq Ch^8 \int_0^t \left(||u||^2_{W^6(I)} + ||u_t||^2_{W^6(I)} \right) d\tau \; , \quad \text{if (4.17a) holds.} \tag{4.24b}$$

It is clear from (4.21) and the bounds (4.22) that

$$\begin{aligned} |z_{xx}|_\partial \leq \frac{Mh^2}{12}|z_{xxt}|_\partial + C\Big(& |||z|||_\infty + |||z_x|||_\infty + h^2|z_{xt}|_\partial \\ & + |T\tilde{e}|_\partial + |R_h|_\partial \Big) \quad , \quad t \in (0,T] \; . \end{aligned} \tag{4.25}$$

Use of (4.23) and (4.25) permits the completion of the estimate (4.18). From (2.2b)

$$\begin{aligned} -\langle z_{xx}, z_t \rangle_A = \; & \frac{1}{2}\frac{d}{dt}\left(||z||^2_{H_0^1(I)} + \frac{h^4}{180}||z_{xxx}||^2_{L^2(I)} \right) \\ & - \frac{h^2}{12} B(z_{xx}, z_{xt}) \; . \end{aligned} \tag{4.26}$$

Observing (via Lemma 2.2) that for any $\varepsilon > 0$

$$\frac{h^2}{12} |B(z_{xx}, z_{xt})| \le \hat{\varepsilon} h^3 |z_{xt}|_\partial^2 + Ch|z_{xx}|_\partial^2 \tag{4.27}$$
$$\le \varepsilon \, \|z_t\|^2_{L^2(I)} + Ch|z_{xx}|_\partial^2 \; ,$$

and adding (4.27) to both sides of (4.18) yields

$$\eta |z_t|^2 + \frac{1}{2} \frac{d}{dt} \left(\|z\|^2_{H^1_o(I)} + \frac{h^4}{180} \|z_{xxx}\|^2_{L^2(I)} \right)$$
$$\le C \left(\|z\|^2_{H^1_o(I)} + h|z_{xx}|^2_\partial + |T\tilde{e}|^2 \right) \tag{4.28}$$
$$+ \varepsilon \|z_t\|^2_{L^2(I)} \; .$$

Integrate (4.28) with respect to t, apply (4.25) to bound the $h|z_{xx}|^2_\partial$ term, and apply Lemma 2.2 to produce

$$\eta \int_0^t |z_t|^2 \, d\tau + \frac{1}{2} \|z\|^2_{H^1_o(I)}(t) \le C \int_0^t \left\{ \|z\|^2_{H^1_o} + |T\tilde{e}|^2 \right.$$
$$\left. + h(|Te|^2_\partial + |R_h|^2_\partial) \right\} d\tau \tag{4.29}$$
$$+ Kh^5 \int_0^t |z_{xxt}|^2_\partial \, d\tau$$
$$+ \left(\varepsilon + \hat{K} h^2 \right) \int_0^t \|z_t\|^2_{L^2(I)} \, d\tau$$
$$+ C \, \|z\|^2_{H^1_o(I)}(0) \; .$$

Now add $(\eta + K)h^5 \int_0^t |z_{xxt}|_\partial^2 \, d\tau$ to (4.29) using the estimate of (4.23). Thus, proceeding as in (4.29)

$$
\begin{aligned}
\eta \int_0^t |||z_t|||^2 \, d\tau + \tfrac{1}{2}||z||^2_{H_o^1(I)}(t) \le C \int_0^t \Big\{ & ||z||^2_{H_o^1(I)} + |T\tilde{e}|^2 \\
& + h\left(|Te|_\partial^2 + |R_h|_\partial^2\right) \Big\} \, d\tau \\
& + (\varepsilon + K^* h^2) \int_0^t ||z_t||^2_{L^2(I)} \, d\tau \\
& + C \, ||z||^2_{H_o^1(I)}(0) \quad .
\end{aligned}
\tag{4.30}
$$

By (2.10) and Gronwall's inequality with ε and h sufficiently small, we obtain our basic estimate

$$\|z_t\|^2_{L^2[0,T;L^2(I)]} + \|z\|^2_{L^\infty[0,T;H^1_o(I)]}$$

$$(4.31)\qquad \leq C\int_0^t \left\{|Te|^2 + h\left(|Te|^2_\partial + |R_h|^2_\partial\right)\right\} d\tau$$

$$+ C\|z\|^2_{H^1_o(I)}(0) \quad .$$

If (1.1) is smooth enough that (4.17a) and (4.20e) hold, then the use of (4.17b), (4.20d), and (4.24b) in (4.31) implies the estimate

$$\|z_t\|^2_{L^2[0,T;L^2(I)]} + \|z\|^2_{L^\infty[0,T;H^1_o(I)]}$$

$$(4.32)\qquad \leq Ch^8\left(\|u_t\|^2_{L^2[0,T;W^6(I)]} + \|u\|^2_{L^2[0,T;W^6(I)]}\right.$$

$$\left. + hK_h^2\right) + C\|z\|^2_{H^1_o(I)}(0) \quad .$$

There are a variety of interpolation schemes which produce $U(x,0)$ such that $\|z\|_{H^1_o(I)}(0) = O(h^4)$. See [15] for several possible choices. Here we choose the interpolation studied in [9]; namely, define $U(x,0)$ by

$$(4.33a)\qquad U(x_i,0) = g(x_i) \quad , \quad 0 \leq i \leq N$$

$$(4.33b)\qquad U_{xx}(x,0) = g''(x) - \frac{h^2}{12} g^{(iv)}(x) \quad , \quad x = 0,1.$$

It is then easy to see (compare with Corollary 3.2) that if $g \in W^6(I)$

$$(4.34)\qquad \|z_{xx}\|_{L^\infty(I)}(0) \leq Ch^4\|g\|_{W^6(I)} \quad .$$

Our final estimate is then

(4.35) $$||z_t||_{L^2[0,T;L^2(I)]} + ||z||_{L^\infty[0,T;H_o^1(I)]} \le Ch^4\left(||u_t||_{L^2[0,T;W^6(I)]} + ||g||_{W^6(I)} + K_h\right),$$

after using the fact that for $\phi \in L^2[0,T;X]$, $\phi(0) \in X$,

$$||\phi||^2_{L^2[0,T;X]} \le C\,||\phi_t||^2_{L^2[0,T;X]} + ||\phi||^2_X\,(0) \quad .$$

The further assumption that $u \in L^\infty[0,T;W^6(I)]$, and Corollary 3.2 imply that

(4.36) $$||\tilde{e}||_{L^\infty[0,T;L^\infty(I)]} \le Ch^4||u||_{L^\infty[0,T;W^6(I)]} \quad .$$

Using (4.35), (4.36), the triangle inequality and the embedding of $H_o^1(I)$ in $L^\infty(I)$; i.e., $||\phi||_{L^\infty(I)} \le C||\phi||_{H_o^1(I)}$, $\phi \in H_o^1(I)$, we obtain the uniform estimate

(4.37) $$||e||_{L^\infty[0,T;L^\infty(I)]} \le Ch^4\left(||u||_{L^\infty[0,T;W^6(I)]} + ||u_t||_{L^2[0,T;W^6(I)]} + K_h\right).$$

<u>Theorem 4.1</u> Suppose that c, c_u, f, f_u, f_{u_x} are uniformly bounded independently of their arguments and that (4.10) and (4.20b,c) hold. Then for h sufficiently small there exists unique U solving (4.1) and (4.33). If u , the solution of (1.1), satisfies

(4.38) $$u \in L^\infty[0,T;W^6(I)] \quad , \quad u_t \in L^2[0,T;W^6(I)] \; ,$$

then

(4.39) $$||u-U||_{L^\infty[0,T;L^\infty(I)]} \le Ch^4\left(||u||_{L^\infty[0,T;W^6(I)]} + ||u_t||_{L^2[0,T;W^6(I)]} + K_h\right).$$

We now show that the use of the simpler boundary collocation (4.1a)' yields $O(h^{7/2})$ estimates; suboptimal in $L^2(I)$ or $L^\infty(I)$ norms, but optimal in $H^1_o(I)$. This loss of accuracy is not believed to be actual; rather, it is just a function of the particular analysis employed. We proceed as before with U defined by (4.1), (4.1a)' and W by (4.12). Then the error analysis is unchanged through (4.18). Note that for $x = 0,1$

(4.41) $$c(W)W_t - W_{xx} = f(u,u_x) + \tilde{e}_{xx}$$

where

(4.42) $$\tilde{e}_{xx}(x,t) = \frac{h^2}{12} u_{xxxx}(x,t) \qquad , \qquad x = 0,1 \quad .$$

Subtract (4.1a)' from (4.41) and use the boundary values to find

(4.43) $$\begin{aligned} -z_{xx} &= f(u,u_x) - f(U,U_x) + \tilde{e}_{xx} \\ &= f^*_{u_x}(z_x + \tilde{e}_x) + \tilde{e}_{xx} \end{aligned}$$

Thus,

(4.44) $$\begin{aligned} -B(z_{xx},z_{xt}) &= B(f^*_{u_x} z_x, z_{xt}) + B(f^*_{u_x} \tilde{e}_x + \tilde{e}_{xx}, z_{xt}) \\ &= \frac{1}{2} B(f^*_{u_x} z_x, \frac{d}{dt} z_x^2) + B(f^*_{u_x} \tilde{e}_x + \tilde{e}_{xx}, z_{xt}) \end{aligned}$$

Integrate by parts

(4.45) $$\begin{aligned} -\int_0^t B(z_{xx},z_{xt})d\tau = &-\frac{1}{2}\int_0^t B\left(\frac{\partial}{\partial t}\left(f^*_{u_x}\right), z_x^2\right)d\tau + \frac{1}{2} B\left(f^*_{u_x}, z_x^2\right)\Big|_o^t \\ &-\int_0^t B\left(\frac{\partial}{\partial t}\left(f^*_{u_x}\tilde{e}_x + \tilde{e}_{xx}\right), z_x\right)d\tau \\ &+ B\left(f^*_{u_x}\tilde{e}_x + \tilde{e}_{xx}, z_x\right)\Big|_o^t \quad . \end{aligned}$$

Now assume that

$$(4.46)\qquad \left|\frac{\partial^2 f}{\partial t\partial u_x}(x,t,\phi,\psi)\right| \le L < \infty\ , \quad x \in I\ ,\ t \in [0,T], \phi,\psi \in \mathcal{R}.$$

Then

$$(4.47)\qquad \begin{aligned} \left|\int_0^t B(z_{xx},z_{xt})d\tau\right| &\le (C+\varepsilon h^{-1})\int_0^t |z_x|_\partial^2\, d\tau + Ch\int_0^t |\tilde{Te}|_\partial^2\, d\tau \\ &+ (C^* + \varepsilon^* h^{-1})\,|z_x|_\partial^2(t) + Ch|\tilde{Re}|_\partial^2\,(t) \\ &+ C(h^{-1}|z_x|_\partial^2 + h|\tilde{Re}|_\partial^2)(0)\ , \end{aligned}$$

where,

$$(4.48a)\qquad |\tilde{Te}|_\partial^2 = |\tilde{e}_x|_\partial^2 + |\tilde{e}_{xx}|_\partial^2 + |\tilde{e}_{xt}|_\partial^2 + |\tilde{e}_{xxt}|_\partial^2$$

and

$$(4.48b)\qquad |Re|_\partial^2 = |\tilde{e}_x|_\partial^2 + |\tilde{e}_{xx}|_\partial^2 \quad .$$

Integrating (4.18) with respect to t and applying (4.26) and (4.46) yields

$$\begin{aligned} \eta\int_0^t |z_t|^2\, d\tau + \frac{1}{2}\,\|z\|^2_{H_o^1(I)}(t) &\le C\int_0^t\Big(\|z\|^2_{H_o^1(I)} + |\tilde{Te}|^2 \\ &+ h^3|\tilde{Te}|_\partial^2\Big)\, d\tau \\ &+ (Ch^2 + \varepsilon h)\int_0^t |z_x|_\partial^2\, d\tau \\ &+ (C^* h^2 + \varepsilon^* h)|z_x|_\partial^2\,(t) \\ &+ Ch^3|\tilde{Re}|_\partial^2\,(t) \\ &+ C\Big(h|z_x|_\partial^2 + h^3|\tilde{Re}|_\partial^2 \\ &+ \|z\|^2_{H_o^1(I)}\Big)(0) \end{aligned}$$

Apply Lemma 2.2 several times and take h so small that $(\frac{1}{2} - C^* h - \varepsilon^*) > 0$.

Then

$$\int_0^t |z_t|^2 \, d\tau + ||z||^2_{H_0^1(I)}(t) \leq C \int_0^t \left(||z||^2_{H_0^1(I)} + |T\tilde{e}|^2 + h^3 |T\tilde{e}|^2_\partial \right) d\tau$$

(4.50)

$$+ C\left(h^3 \left(\max_{0 \leq \tau \leq t} |R\tilde{e}|^2_\partial \right) + ||z||^2_{H_0^1(I)}(0) \right).$$

Gronwall's inequality implies that

$$\int_0^T |z_t|^2 \, d\tau + ||z||^2_{L^\infty[0,T;H_0^1(I)]}$$

(4.51)

$$\leq C \int_0^T \left(|T\tilde{e}|^2 + h^3 |T\tilde{e}|^2_\partial \right) d\tau$$

$$+ C\left(h^3 \left(\max_{0 \leq t \leq T} |Re|^2_\partial \right) + ||z||^2_{H_0^1(I)}(0) \right).$$

Note that

(4.52) $$h^3 |\tilde{e}_{xx}|^2_\partial = \frac{h^7}{144} |u_{xxxx}|^2_\partial \quad ; \quad h^3 |\tilde{e}_{xxt}|^2_\partial = \frac{h^7}{144} |u_{xxxxt}|^2_\partial .$$

Hence, under assumption (4.17a) and the choice of $U(x,0)$ given in (4.33), we obtain

$$\int_0^T |z_t|^2 \, d\tau + ||z||^2_{L^\infty[0,T;H_0^1(I)]}$$

(4.53)

$$\leq Ch^8 \left(||u_t||^2_{L^2[0,T;W^6(I)]} + ||g||^2_{W^6(I)} \right)$$

$$+ Ch^7 \left(\int_0^T |u_{xxxxt}|^2_\partial \, d\tau + \max_{0 \leq \tau \leq T} |u_{xxxx}|^2_\partial \right).$$

This estimate leads easily to the optimal $O(h^3)$ estimate in the $H_o^1(I)$ norm; however, it implies only $O(h^{7/2})$ estimate in the $L^2(I)$ or $L^\infty(I)$ norm. Of course if

$$u_{xxxx}(x,t) = 0 \ , \quad x = 0,1 \ , \quad t \in [0,T] \ , \tag{4.54}$$

the estimate becomes $O(h^4)$. Similarly, if u is periodic, i.e.,

$$D_x^j u(0,t) = D_x^j u(1,t) \ , \quad 0 \le j \le 2 \ , \quad t \in [0,T] \ , \tag{4.55}$$

then making U similarly periodic results in $O(h^4)$ estimates since $B(z_{xx}, z_{xt}) = 0$. Note that in this case (4.1a) is replaced by (4.55) as applied to U.

<u>Theorem 4.2</u> Under the assumptions of Theorem 4.1, with (4.10)' and (4.46) replacing (4.10) and (4.20), let U be defined by (4.1) with (4.1a)' replacing (4.1a). Then,

$$||u-U||_{L^\infty[0,T;H_o^1(I)]} \le Ch^3 \left(||u||_{L^\infty[0,T;W^6(I)]} + ||u_t||_{L^2[0,T;W^6(I)]} \right) , \tag{4.56a}$$

$$||u-U||_{L^\infty[0,T;L^\infty(I)]} \le Ch^4 \left(||u||_{L^\infty[0,T;W^6(I)]} + ||u_t||_{L^2[0,T;W^6(I)]} \right) + Ch^{7/2} \left(\max_{0\le t\le T} |u_{xxxx}|_\partial + \left(\int_0^T |u_{xxxxt}|_\partial^2 \right)^{1/2} \right) . \tag{4.56b}$$

If, in addition, (4.54) holds

$$||u-U||_{L^\infty[0,T;L^\infty(I)]} \le Ch^4 \left(||u||_{L^\infty[0,T;W^6(I)]} + ||u_t||_{L^2[0,T;W^6(I)]} \right) . \tag{4.56c}$$

Furthermore, if (4.55) holds, and U is also required to be periodic, then (4.56c) obtains.

REFERENCES

1. E.L.Albasiny and W.D.Hoskins, Increased accuracy cubic spline solutions to two-point boundary value problems, J.Inst.Maths.Applics., 9(1972), 47-55.

2. D.A.Archer, Cubic spline collocation methods for nonlinear parabolic problems, contributed:Fall SIAM-SIGNUM meeting, Austin,Tx., Oct. 1972.

3. __________, Some collocation methods for differential equations, Ph.D. Thesis, Rice Univ., Houston, Tx., 1973.

4. C.R.deBoor, The method of projections as applied to the numerical solution of two point boundary value problems using cubic splines, Ph.D.Thesis, Univ. of Michigan, Ann Arbor, 1966.

5. __________, On calculating with B-splines, J.Approx. Thy.,6(1972), 50-62.

6. __________, Package for calculating with B-splines,to appear SIAM J. Num. Anal.

7. __________, B.K.Swartz, Collocation at Gaussian points, SIAM J. Num.Anal., 10(1972),582-606.

8. J.C.Cavendish, Collocation methods for elliptic and parabolic boundary value problems, Ph.D.Thesis, Univ. of Pittsburg, 1972.

9. J.W.Dainel and B.K.Swartz, Extrapolated collocation for two-point boundary value probelms using cubic splines, Los Alamos Scientific Laboratory Technical Report LA-DC-72-1520, Dec.1972.

10. J.Douglas,Jr., and T.Dupont, A finite element collocation method for quasilinear parabolic equations, Math. Comp., 27(1973), 17-28.

11. __________,__________, Collocation methods for parabolic equations in a single space variable, Springer-Verlag, Berlin, 1974.

12. D.J.Fyfe, The use of cubic splines in the solution of two-point boundary value problems, Computer Journal, 12(1969), 188-192.

13. R.S.Hirsh, Application of a fourth order differencing technique to fluid mechanics problems, contributed: Fall SIAM meeting, Alexandria, Va., Oct. 1974.

14. J.L.Lions and E.Magenes, Non-homogeneous boundary value probelms and applications, Springer-Verlag, New York, 1972.

15. T.R.Lucas, Error bounds for interpolating cubic splines under various end conditions, SIAM J. Num.Anal., 11(1974),569-584.

16. __________, G.W.Reddien, Some collocation methods for nonlinear boundary value problems, SIAM J.Num.Anal., 9(1972),341-356

17. __________,__________, A high order projection method for nonlinear two point boundary value problems, Numer.Math., 20(1973),257-270.

18. N.Papamichael and J.R.Whiteman, A cubic spline technique for the one dimensional heat conduction problem, J.Inst.Maths.Applics., 11(1973), 111-113.

19. S.G.Rubin and R.A.Graves, A cubic spline approximation for problems in fluid mechanics, School of Engineering Old Dominion Univ. Technical Report 74-T1, Norfolk ,Va.,June 1974.

20. R.D.Russel and L.F.Shampine, A collocation method for boundary value problems, Numer.Math., 19(1972),1-28.

21. B.K.Swartz, $O(h^{k-j}\omega(D^k f,h))$ bounds on some spline interpolation errors, Los Alamos Scientific Laboratory Technical Report, LA-4477,1970

22. __________, R.S.Varga, Error bounds for spline and L-spline interpolation, J.Approx.Thy., 6(1972),6-49.

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